

# A second update on double parton distributions

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We present two equivalent consistency checks of the momentum sum rule for double parton distributions and show the importance of the inclusion of the so-called inhomogeneous term in order to preserve correct longitudinal momentum correlations. We further discuss in some detail the kinematics of the splitting at the basis of the inhomogeneous term and update the double parton distributions evolution equations at different virtualities.

Keywords: Double parton distributions, QCD evolution equations, QCD sum rules

## I. INTRODUCTION

The hadron internal structure is presently encoded, thanks to the QCD factorisation theorem, in process-independent parton distribution functions (PDFs). The latter allow to predict cross sections for high-mass systems and high transverse-momentum jets in hadronic collisions in terms of binary partonic interactions. There are, however, increasing experimental evidences (for recent analyses see Ref. [1]) that hard double parton scattering (DPS) may occur within the same hadronic collision. The experimental and theoretical efforts to identify and quantify DPS contributions aim to understand and control this additional QCD background in new physics searches, especially in the multi-jet channel. At a more fundamental level, DPS could unveil parton correlations in the hadron structure not accessible in single parton scattering (SPS) and encoded in novel distributions, *i.e.* double parton distributions (DPDs). So far, measurements have only provided informations on  $\sigma_{eff}$ . This dimensionful parameter controls the magnitude of DPS contribution under the simplifying assumptions of two uncorrelated hard scatterings and full factorisation of DPDs in terms

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of ordinary PDFs and model-dependent distribution in transverse position space. Many theoretical analyses have predicted QCD evolution effects on DPDs relaxing some or all the above assumptions [2–6]. A part recent progress in this direction reported in Ref. [7], the experimental observation of the expected mild scaling violations induced by DPDs evolution is not yet possible given the accuracy of the present data. Nonetheless, a good theoretical control of the latter is mandatory if the whole DPS formalism has to be properly validated against data. A first attempt to calculate the scale dependence of longitudinal DPDs (hereafter called IDPDs) has been presented long ago in Ref. [2] under the assumption of factorisation in transverse space. With respect to standard single-parton distributions [8], IDPDs evolution equations do contain an additional term which is responsible for perturbative longitudinal correlation between the interacting partons. This result has stimulated in the recent past an increasing activity in the field and has generated some constructive criticism in the literature. A first critical point is that the relative transverse momentum of the interacting parton pair is not conserved between amplitude and its conjugated [9]. This implies that one should consider new distributions, addressed as two-particle generalised parton distributions,  $_2\text{GPDs}$ , which have an additional dependence on a transverse momentum vector  $\Delta$  which parametrises this imbalance [9]. They reduce to IDPDs addressed in this paper when this vector is set to zero or, in position space, if they are integrated over the relative distance  $b$  of the parton pair. This additional dependence affects the evolution of the correlated and uncorrelated terms in rather different way [6] and give rise to inconsistencies with respect to the formalism of Refs. [2, 3]. More importantly,  $_2\text{GPDs}$  enter the DPS cross sections rather than their  $b$ -integrated or  $\Delta = 0$  counterparts, *i.e.* longitudinal DPDs, and moreover the integral over the imbalance  $\Delta$  of the product of  $_2\text{GPDs}$  is directly proportional to the value of  $\sigma_{eff}^{-1}$  [9, 10]. A second critical point is that the inclusion of single splitting contributions, according to the formalism of Ref. [2], poses a problem of consistency with SPS loop corrections when DPDs are used to evaluate DPS cross sections. A problem which is solved if one considers two-particle generalised parton distributions,  $_2\text{GPDs}$  [11]. From these observations, it appears that  $_2\text{GPDs}$  offer a natural solution to this class of problems and are a good candidate to focus on when addressing the issues related to QCD evolution. On the other hand, as we shall describe in the following, the presence of the inhomogeneous term in the evolution equations appearing Ref. [2] is crucial if one demands that longitudinal DPDs satisfy QCD consistency check for the momentum sum rule. It appears therefore that

the road towards a consistent treatment of QCD evolution effects on DPDs is quite narrow as it must reconcile all these requirements at once.

This paper is organized as follows. In Sec. II we collect some definitions and formulas pertinent to the Jet Calculus formalism [12] and frequently used thereafter. In Sec. III we present two equivalent derivations of the momentum sum rule for IDPDs, paying particular attention to some delicate steps occurring in the calculation. In Sec. IV we discuss in some detail the kinematics of the splitting in the inhomogeneous term and update the IDPDs evolution equations at different virtualities in light of the results obtained for the momentum sum rule. We summarise our results in Sec. V.

## II. PRELIMINARIES

We recall briefly the main ingredients which we will use in our calculations. The longitudinal double-parton distributions  $D_h^{j_1, j_2}(x_1, Q_1^2, x_2, Q_2^2)$  are interpreted as the two-particle inclusive distribution to find in a target hadron a couple of partons of flavour  $j_1$  and  $j_2$  with fractional momenta  $x_1$  and  $x_2$  and virtualities up to  $Q_1^2$  and  $Q_2^2$ , respectively. The distributions at the final scales,  $Q_1^2$  and  $Q_2^2$ , are constructed through the parton-to-parton functions,  $E$ , which themselves obey DGLAP-type [8] evolution equations:

$$Q^2 \frac{\partial}{\partial Q^2} E_i^j(x, Q_0^2, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{du}{u} P_k^i(u) E_i^k(x/u, Q_0^2, Q^2), \quad (1)$$

with initial condition  $E_i^j(x, Q_0^2, Q_0^2) = \delta_i^j \delta(1-x)$  and  $P_k^i(u)$  the Altarelli-Parisi splitting functions. The functions  $E$  provide the resummation of collinear logarithms up to the accuracy with which the  $P_k^i(u)$  are specified. We may therefore express  $D_h^{j_1, j_2}(x_1, Q_1^2, x_2, Q_2^2)$  as

$$D_h^{j_1, j_2}(x_1, Q_1^2, x_2, Q_2^2) = \int_{x_1}^{1-x_2} \frac{dz_1}{z_1} \int_{x_2}^{1-z_1} \frac{dz_2}{z_2} \left[ D_h^{j'_1, j'_2}(z_1, Q_0^2, z_2, Q_0^2) E_{j'_1}^{j_1}\left(\frac{x_1}{z_1}, Q_0^2, Q_1^2\right) E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, Q_0^2, Q_2^2\right) + \int_{Q_0^2}^{\text{Min}(Q_1^2, Q_2^2)} d\mu_s^2 D_{h, \text{corr}}^{j'_1, j'_2}(z_1, z_2, \mu_s^2) E_{j'_1}^{j_1}\left(\frac{x_1}{z_1}, \mu_s^2, Q_1^2\right) E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, \mu_s^2, Q_2^2\right) \right]. \quad (2)$$

The first term on r.h.s., usually addressed as the homogeneous term, takes into account the uncorrelated evolution of the active partons found at a scale  $Q_0^2$  in  $D_h^{j'_1, j'_2}$  up to  $Q_1^2$  and  $Q_2^2$ , respectively. The second term, the so-called inhomogeneous one, takes into account the

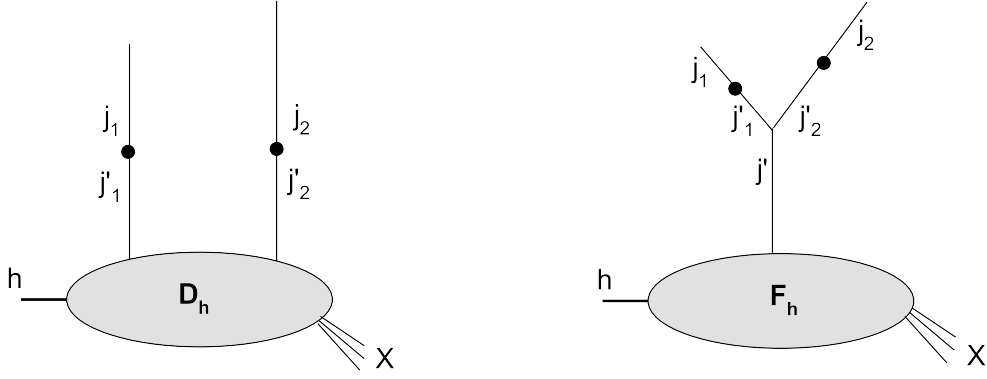


FIG. 1: Pictorial representation of both terms on right hand side of eq. (2). Black dots symbolize the parton-to-parton evolution function,  $E$ .

probability to find the active partons at  $Q_1^2$  and  $Q_2^2$  as a result of a splitting at a scale  $\mu_s^2$ , integrated over all the intermediate scale at which such splitting may occur. The distribution  $D_{h,corr}^{j'_1,j'_2}$  is given by

$$D_{h,corr}^{j'_1,j'_2}(z_1, z_2, \mu_s^2) = \frac{\alpha_s(\mu_s^2)}{2\pi\mu_s^2} \frac{F_h^{j'}(z_1 + z_2, \mu_s^2)}{z_1 + z_2} \hat{P}_{j'}^{j'_1,j'_2}\left(\frac{z_1}{z_1 + z_2}\right). \quad (3)$$

In eq. (3),  $F_h^{j'}$  are single parton distributions and  $\hat{P}_{j'}^{j'_1,j'_2}$  are the real Altarelli-Parisi splitting functions [12]. Both terms in eq. (2) are shown in Fig. (1). The scale  $Q_0^2$  is in general the (low) scale at which IDPDs are usually modelled, in complete analogy with the single-parton distributions case. In the present context it also acts as the factorisation scale for the correlated term, since all unresolved splittings, for which  $\mu_s^2 < Q_0^2$ , are effectively taken into account in the parametrisation of  $D_h^{j'_1,j'_2}(z_1, Q_0^2, z_2, Q_0^2)$ . In the “equal scales” case, taking the logarithmic derivative with respect to  $Q^2$  in eq. (2), we recover the result presented in Ref. [2]:

$$Q^2 \frac{\partial D_h^{j_1,j_2}(x_1, x_2, Q^2)}{\partial Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_{\frac{x_1}{1-x_2}}^1 \frac{du}{u} P_k^{j_1}(u) D_h^{j_2,k}(x_1/u, x_2, Q^2) + \frac{\alpha_s(Q^2)}{2\pi} \int_{\frac{x_2}{1-x_1}}^1 \frac{du}{u} P_k^{j_2}(u) D_h^{j_1,k}(x_1, x_2/u, Q^2) + \frac{\alpha_s(Q^2)}{2\pi} \frac{F_h^{j'}(x_1 + x_2, Q^2)}{x_1 + x_2} \hat{P}_{j'}^{j_1,j_2}\left(\frac{x_1}{x_1 + x_2}\right). \quad (4)$$

The first and second terms on the right-hand side are obtained through the  $Q^2$  dependence contained in the  $E$  functions, while the last is obtained from the  $Q^2$  dependent limit in the  $\mu_s^2$  integration in the correlated term. The IDPDs evolution equations therefore resum large

contributions of the type  $\alpha_s \ln(Q^2/Q_0^2)$  and  $\alpha_s \ln(Q^2/\mu_s^2)$  appearing in the uncorrelated and correlated term of eq. (2), respectively.

### III. MOMENTUM SUM RULE

A number of sum rules for IDPDs has been already discussed and used to constrain initial conditions for IDPDs evolution in Ref. [13]. Sum rules are in general expected to hold on the basis of unitarity of the relevant cross sections [14]. In the following we show that the momentum sum rule for DPDs satisfies the necessary, but not sufficient for it to hold, condition of being preserved under QCD evolution. For this purpose we assume that the momentum sum rule is valid at an arbitrary but still perturbative scale  $Q_0^2 < Q_1^2, Q_2^2$ :

$$\sum_{j'_1} \int_0^{1-z_2} dz_1 z_1 D_h^{j'_1, j'_2}(z_1, Q_0^2, z_2, Q_0^2) = (1 - z_2) F_h^{j'_2}(z_2, Q_0^2), \quad (5)$$

which, as customary, we choose as the starting scale for evolution. The aim of the calculation is therefore to verify that, once imposed at  $Q_0^2$ , DPDs fulfil the momentum sum rule

$$\sum_{j_1} \int_0^{1-x_2} dx_1 x_1 D_h^{j_1, j_2}(x_1, Q_1^2, x_2, Q_2^2) = (1 - x_2) F_h^{j_2}(x_2, Q_2^2), \quad (6)$$

at any other scales  $Q_1^2$  and  $Q_2^2$ , which we keep deliberately different. This strategy has been used to check an analogous sum rule in the context of dihadron fragmentation functions [12] and fracture functions [15] while for IDPDs an explicit calculation has been presented in some detail in Ref. [4]. We apply first the sum rule, eq. (6), to the homogeneous term in eq. (2)

$$\sum_{j_1} \int_0^{1-x_2} dx_1 x_1 \int_{x_1}^{1-x_2} \frac{dz_1}{z_1} \int_{x_2}^{1-z_1} \frac{dz_2}{z_2} D_h^{j'_1, j'_2}(z_1, Q_0^2, z_2, Q_0^2) E_{j'_1}^{j_1}\left(\frac{x_1}{z_1}, Q_0^2, Q_1^2\right) E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, Q_0^2, Q_2^2\right). \quad (7)$$

We note that, reordering the  $z_1$  and  $x_1$  integrals, the function  $E_{j'_1}^{j_1}$  can be evaluated through the momentum sum rule for  $E$ -functions which reads

$$\sum_{j_1} \int_0^1 dz z E_{j'_1}^{j_1}(z, Q_0^2, Q_1^2) = 1. \quad (8)$$

This property is derived in the appendix A and it might be thought as the parton level analogue of the momentum sum rule for fragmentation functions. Quite importantly, by

using eq. (8), any explicit dependence on  $Q_1^2$  disappears and we obtain

$$\int_0^{1-x_2} dz_1 z_1 \sum_{j'_1} \int_{x_2}^{1-z_1} \frac{dz_2}{z_2} D_h^{j'_1, j'_2}(z_1, Q_0^2, z_2, Q_0^2) E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, Q_0^2, Q_2^2\right). \quad (9)$$

Reordering again the integrals we get

$$\int_{x_2}^1 \frac{dz_2}{z_2} \left[ \sum_{j'_1} \int_0^{1-z_2} dz_1 z_1 D_h^{j'_1, j'_2}(z_1, Q_0^2, z_2, Q_0^2) \right] E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, Q_0^2, Q_2^2\right), \quad (10)$$

where we now recognise in square brackets the sum rule at  $Q_0^2$  in eq. (5). For the homogeneous term we therefore obtain the following contribution

$$\int_{x_2}^1 \frac{dz_2}{z_2} (1-z_2) F_h^{j'_2}(z_2, Q_0^2) E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, Q_0^2, Q_2^2\right) \quad (11)$$

and conclude that its contribution alone does not reconstruct the expected result in eq. (6).

We now turn to the inhomogeneous term in eq. (2). Applying eq. (6) we get

$$\begin{aligned} & \sum_{j_1} \int_0^{1-x_2} dx_1 x_1 \int_{x_1}^{1-x_2} \frac{dz_1}{z_1} \int_{x_2}^{1-z_1} \frac{dz_2}{z_2} \\ & \cdot \int_{Q_0^2}^{Q_M^2} d\mu_s^2 \frac{\alpha_s(\mu_s^2)}{2\pi\mu_s^2} \frac{F_h^{j'}(z_1+z_2, \mu_s^2)}{z_1+z_2} \widehat{P}_{j'_1, j'_2}^{j_1, j_2}\left(\frac{z_1}{z_1+z_2}\right) E_{j'_1}^{j_1}\left(\frac{x_1}{z_1}, \mu_s^2, Q_1^2\right) E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, \mu_s^2, Q_2^2\right). \end{aligned} \quad (12)$$

As in the previous case, the  $E$ -function which describes the evolution of the first parton can be integrated out by using  $E$ -momentum sum rule, eq. (8). Changing then variable to  $u = z_1/(z_1+z_2)$  we obtain

$$\int_{Q_0^2}^{Q_M^2} d\mu_s^2 \frac{\alpha_s(\mu_s^2)}{2\pi\mu_s^2} \int_{x_2}^1 \frac{dz_2}{z_2} \int_{z_2}^1 \frac{du}{u} \frac{1-u}{u} F_h^{j'}\left(\frac{z_2}{u}, \mu_s^2\right) \sum_{j'_1} \widehat{P}_{j'_1, j'_2}^{j_1, j_2}(1-u) E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, \mu_s^2, Q_2^2\right). \quad (13)$$

In order to proceed we need to relate real and regularised splitting functions. A number of these relations, valid in Mellin moment space, have been worked out in Ref. [12]. The following relation is needed for our purpose

$$\int_x^1 du (1-u) g(u) \sum_{j_1} \widehat{P}_{j_1}^{j_1, j_2}(1-u) = \int_x^1 du (1-u) g(u) P_{j_1}^{j_2}(u), \quad (14)$$

where  $g(u)$  is a regular function of  $u$ . Since real and regularised splitting functions differ by the inclusion of virtual terms, proportional to  $\delta(1-u)$  in the latter, the relation holds only at the integral level. It generalises the symmetry of real splitting functions upon the exchange of daughter partons

$$\widehat{P}_{j'}^{j_1, j_2}(1-u) = \widehat{P}_{j'}^{j_2, j_1}(u). \quad (15)$$

Since this relation is not often encountered in the literature we provide its explicit evaluation in appendix B. With the help of eq. (14) and eq. (15) we obtain

$$\int_{Q_0^2}^{Q_M^2} d\mu_s^2 \frac{\alpha_s(\mu_s^2)}{2\pi\mu_s^2} \int_{x_2}^1 \frac{dz_2}{z_2} \left[ \int_{z_2}^1 \frac{du}{u^2} - \int_{z_2}^1 \frac{du}{u} \right] F_h^{j'}\left(\frac{z_2}{u}, \mu_s^2\right) P_{j'}^{j'_2}(u) E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, \mu_s^2, Q_2^2\right), \quad (16)$$

where we have split the  $u$ -integral in square brackets in two terms, A and B, respectively. In order to deal with the  $\mu_s^2$ -integral, we need to build up a full derivative term with respect to  $\mu_s^2$  out of the two  $F \otimes P \otimes E$  terms in eq. (16). For this purpose, we write  $F_h^{j'}$  as a solution of its evolution equation

$$F_h^{j'}\left(\frac{z_2}{u}, \mu_s^2\right) = \int_{\frac{z_2}{u}}^1 \frac{dw}{w} F_h^r(w, Q_0^2) E_r^{j'}\left(\frac{z_2}{uw}, Q_0^2, \mu_s^2\right), \quad (17)$$

and substitute it in eq. (16). For the B-term we get

$$B = - \int_{Q_0^2}^{Q_M^2} d\mu_s^2 \frac{\alpha_s(\mu_s^2)}{2\pi\mu_s^2} \int_{x_2}^1 \frac{dz_2}{z_2} \int_{z_2}^1 \frac{du}{u} \int_{\frac{z_2}{u}}^1 \frac{dw}{w} F_h^r(w, Q_0^2) E_r^{j'}\left(\frac{z_2}{uw}, Q_0^2, \mu_s^2\right) \cdot P_{j'}^{j'_2}(u) E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, \mu_s^2, Q_2^2\right). \quad (18)$$

The  $E_r^{j'} P_{j'}^{j'_2}$  term is a convolution over the  $u$ -variable and has the basic structure of the r.h.s. of eq. (1), so it can be rewritten as a  $\mu_s^2$ -derivative:

$$B = - \int_{Q_0^2}^{Q_M^2} d\mu_s^2 \int_{x_2}^1 dz_2 \int_{z_2}^1 \frac{dw}{w} F_h^r(w, Q_0^2) \left[ \frac{\partial}{\partial \mu_s^2} E_r^{j'_2}\left(\frac{z_2}{w}, Q_0^2, \mu_s^2\right) \right] E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, \mu_s^2, Q_2^2\right). \quad (19)$$

Focusing now on the A-term, we would like to proceed as in the previous case and build a  $\mu_s$ -derivative out of the term  $P_{j'}^{j'_2} E_{j'_2}^{j_2}$ . In this case, however, the convolution variable and the matrix structure of the product do not allow a direct use of eq. (1). In order to bring this term in a more manageable form, we first reorder the integrals and then change variables to the new convolution variable  $y = x_2 u / z_2$ . We get

$$A = \int_{Q_0^2}^{Q_M^2} d\mu_s^2 \frac{\alpha_s(\mu_s^2)}{2\pi\mu_s^2} \int_{x_2}^1 \frac{dw}{w} F_h^r(w, Q_0^2) \int_{x_2/w}^1 \frac{dy}{y^2} x_2 E_r^{j'}\left(\frac{x_2}{yw}, Q_0^2, \mu_s^2\right) \cdot \int_y^1 \frac{du}{u} P_{j'}^{j'_2}(u) E_{j'_2}^{j_2}\left(\frac{y}{u}, \mu_s^2, Q_2^2\right). \quad (20)$$

We notice that now the matrix structure of the term  $P_{j'}^{j'_2} E_{j'_2}^{j_2}$  does correspond to the right-hand side of eq. (1) transposed. Taking this into account, we may rewrite it as a  $\mu_s^2$ -derivative:

$$A = \int_{Q_0^2}^{Q_M^2} d\mu_s^2 \int_{x_2}^1 \frac{dw}{w} F_h^r(w, Q_0^2) \int_{x_2/w}^1 \frac{dy}{y^2} x_2 E_r^{j'}\left(\frac{x_2}{yw}, Q_0^2, \mu_s^2\right) \left[ - \frac{\partial}{\partial \mu_s^2} E_{j'}^{j_2}\left(y, \mu_s^2, Q_2^2\right) \right], \quad (21)$$

where the extra minus sign comes from having differentiated the  $E$ -function with respect to the lower scale. If now we change back to  $z_2 = x_2/y$ , the A and B term have the same integral structure and can be summed together giving

$$A+B = - \int_{Q_0^2}^{Q_M^2} d\mu_s^2 \int_{x_2}^1 \frac{dw}{w} F_h^r(w, Q_0^2) \int_{x_2}^w dz_2 \frac{\partial}{\partial \mu_s^2} \left[ E_r^{j'} \left( \frac{z_2}{w}, Q_0^2, \mu_s^2 \right) E_{j'}^{j_2} \left( \frac{x_2}{z_2}, \mu_s^2, Q_2^2 \right) \right]. \quad (22)$$

Now the  $\mu_s^2$  can be performed trivially and by using the initial conditions for the  $E$ -functions we obtain

$$A + B = - \int_{x_2}^1 \frac{dw}{w} F_h^r(w, Q_0^2) \int_{x_2}^w dz_2 \left[ E_r^{j'} \left( \frac{z_2}{w}, Q_0^2, Q_2^2 \right) \delta_{j'}^{j_2} \delta \left( 1 - \frac{x_2}{z_2} \right) - \delta_r^{j'} \delta \left( 1 - \frac{z_2}{w} \right) E_{j'}^{j_2} \left( \frac{x_2}{z_2}, Q_0^2, Q_2^2 \right) \right]. \quad (23)$$

Simplifying and changing  $w$  to  $z_2$  we get

$$A + B = \int_{x_2}^1 \frac{dz_2}{z_2} F_h^r(z_2, Q_0^2) (z_2 - x_2) E_r^{j_2} \left( \frac{x_2}{z_2}, Q_0^2, Q_2^2 \right). \quad (24)$$

Adding the result coming from the homogeneous calculation, eq. (11), we get

$$(1 - x_2) \int_{x_2}^1 \frac{dz_2}{z_2} F_h^r(z_2, Q_0^2) E_r^{j_2} \left( \frac{x_2}{z_2}, Q_0^2, Q_2^2 \right). \quad (25)$$

By using eq. (17) we may rewrite the above convolution simply as  $F_h^{j_2}(x_2, Q_2^2)$  and finally obtain the desired momentum sum rule in eq. (6). The large number of convolution integrals involved renders however such calculation not really transparent. There is, however, an easier way to obtain the same result, that is to apply eq. (6) directly to the IDPDs evolution equations, eq. (4), and verify that, with this procedure, one recovers single PDFs evolution. We will present this calculation in the “equal scales” case. When the sum rules operates on the left hand side of eq. (4), it simply gives the scale derivative of ordinary parton distributions weighted by the factor  $(1 - x_2)$ :

$$(1 - x_2) Q^2 \frac{\partial}{\partial Q^2} F_h^{j_2}(x_2, Q^2). \quad (26)$$

Applying the momentum sum rule the first term on the right hand side of eq. (4), which corresponds to the uncorrelated evolution of the first parton, we obtain

$$\frac{\alpha_s(Q^2)}{2\pi} \sum_{j_1} \int_0^{1-x_2} dx_1 x_1 \int_{\frac{x_1}{1-x_2}}^1 \frac{du}{u} P_k^{j_1}(u) D_h^{j_2, k}(x_1/u, x_2, Q^2). \quad (27)$$



Reordering the integrals and changing variable to  $y = x_1/u$  we get

$$\frac{\alpha_s(Q^2)}{2\pi} \sum_{j_1} \int_0^1 du u P_k^{j_1}(u) \int_0^{1-x_2} dy y D_h^{k,j_2}(y, x_2, Q^2). \quad (28)$$

Exploiting now the basic property of splitting functions

$$\sum_{j_1} \int_0^1 du u P_k^{j_1}(u) = 0, \quad (29)$$

the term corresponding to the evolution of parton first vanishes, i.e. the overall momentum carried by the first parton is a quantity conserved by evolution. We now apply the sum rule to the second term of eq. (4), which corresponds to the uncorrelated evolution of the second parton and obtain

$$\frac{\alpha_s(Q^2)}{2\pi} \sum_{j_1} \int_0^{1-x_2} dx_1 x_1 \int_{\frac{x_2}{1-x_1}}^1 \frac{du}{u} P_k^{j_2}(u) D_h^{j_1,k}(x_1, x_2/u, Q^2). \quad (30)$$

Reordering the integrals we obtain

$$\frac{\alpha_s(Q^2)}{2\pi} \int_{x_2}^1 \frac{du}{u} P_k^{j_2}(u) \left[ \sum_{j_1} \int_0^{1-x_2/u} dx_1 x_1 D_h^{j_1,k}(x_1, x_2/u, Q^2) \right]. \quad (31)$$

We recognise in square brackets the momentum sum rule written for momentum fraction  $x_1$  and  $x_2/u$  so that eq. (31) becomes

$$\frac{\alpha_s(Q^2)}{2\pi} \int_{x_2}^1 \frac{du}{u} P_k^{j_2}(u) \left[ 1 - \frac{x_2}{u} \right] F_h^k\left(\frac{x_2}{u}, Q^2\right). \quad (32)$$

We interpret the term in square brackets as the fractional momentum of the proton (1) minus the fractional momentum of the second parton before evolution,  $x_2/u$ . We finally handle the correlated term in the evolution equations,

$$\sum_{j_1} \int_0^{1-x_2} dx_1 x_1 \frac{\alpha_s(Q^2)}{2\pi} \frac{F_h^{j'}(x_1 + x_2, Q^2)}{x_1 + x_2} \widehat{P}_{j'}^{j_1, j_2}\left(\frac{x_1}{x_1 + x_2}\right). \quad (33)$$

We change variables to  $u = x_2/(x_1 + x_2)$  and get

$$\frac{\alpha_s(Q^2)}{2\pi} \int_{x_2}^1 \frac{du}{u} x_2 \frac{1-u}{u} F_h^{j'}\left(\frac{x_2}{u}, Q^2\right) \sum_{j_1} \widehat{P}_{j'}^{j_1, j_2}(1-u). \quad (34)$$

With the help of eq. (14) and eq. (15), we obtain

$$\frac{\alpha_s(Q^2)}{2\pi} \int_{x_2}^1 \frac{du}{u} \left[ \frac{x_2}{u} - x_2 \right] F_h^{j'}\left(\frac{x_2}{u}, Q^2\right) P_{j'}^{j_2}(u). \quad (35)$$

Again it is interesting to interpret the result and to give an explanation to the factor  $x_2/u - x_2$ . The momentum fraction of the (second) interacting parton prior to the branching is  $x_2/u$  while after the branching it has a fixed momentum fraction  $x_2$ . Therefore the quantity in square brackets is simply the fractional momentum of the first, integrated-over, parton. If we now sum the results in eq. (32) and eq. (31), the  $u$ -dependent terms cancel each other, and the net result is

$$\frac{\alpha_s(Q^2)}{2\pi} \int_{x_2}^1 \frac{du}{u} P_k^{j_2}(u) [1 - x_2] F_h^k\left(\frac{x_2}{u}, Q^2\right). \quad (36)$$

Equating this result to eq. (26), the factor  $1 - x_2$  can be simplified on both side and we obtain the familiar sPDF evolution equation for the second parton [8]:

$$Q^2 \frac{\partial}{\partial Q^2} F_h^{j_2}(x_2, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int_{x_2}^1 \frac{du}{u} P_k^{j_2}(u) F_h^k\left(\frac{x_2}{u}, Q^2\right). \quad (37)$$

From both calculations, it is clear the crucial role played by the inhomogeneous term in order that the momentum sum rule for DPDs is preserved under QCD evolution. When the latter is directly applied to it, it takes into account the amount of fractional momentum lost by the second parton due to perturbative parton emissions. Or in other words, the contributions to the momentum sum rule coming from initial state radiation. If such term is neglected altogether, the consistency of the formalism is lost since longitudinal momentum correlations are not properly taken into account. For these reasons, checking the sum rule is a useful method to investigate the consistency of the evolution equations.

#### IV. KINEMATIC OF THE SPLITTING TERM

In a previous paper [3] we have proposed the DPDs evolution equations at different virtualities. This case is potentially relevant since many experimental DPS analyses consider the associated production of an electroweak boson,  $Q_2^2 \simeq M_{W^\pm, Z}^2$ , with jets,  $Q_1^2 \simeq P_T^2$ , where  $P_T$  is the jet transverse momentum typically chosen to be larger than 15 GeV at LHC. Since DPS contributions are expected to populate low- $p_t$  particle spectrum, one may trigger on identified particles rather than on jets. This case, in which again  $Q_2^2 \gg Q_1^2$ , has been considered in Ref. [17] for the associated production of a  $Z$ -boson and a  $D$ -meson in the forward region of pp collisions at LHC and whose SPS background can be evaluated by using the results presented in Refs. [18]. Turning back to the DPDs evolution equations at

different virtualities and, to be definite, considering the case  $Q_1^2 < Q_2^2$ , we have found that the proposed homogeneous evolution equations with respect to the higher scale  $Q_2^2$  does not fulfil the momentum sum rule, eq. (6). The disappearance of the inhomogeneous term was caused by the choice of the upper limit of the  $\mu_s^2$  integral which, from strong ordering of virtualities implied by leading logarithmic approximation, was chosen to be  $\text{Min}(Q_1^2, Q_2^2)$ , and therefore independent of  $Q_2^2$ . This fact has induced us to reconsider in more detail the kinematics of the branching at the basis of the inhomogeneous term. We first note that a physically plausible scale characterising the parton branching in the inhomogeneous term could be identified with the relative transverse momentum squared,  $\mathbf{r}_T^2$ , between the daughter partons, rather than the generic  $\mu_s^2$  scale. This scale choice ambiguity can be resolved only within a higher order calculation. We can easily calculate this quantity considering a generic branching  $p_0(1, \mathbf{0}_T, p_0^2) \rightarrow p_1(z, \mathbf{r}_T, p_1^2) + p_2(1-z, -\mathbf{r}_T, p_2^2)$ , where we have explicitly indicated longitudinal momentum fractions, transverse momenta and space-like virtualities of the relevant partons. By performing a Sudakov decomposition of the four-momenta and setting  $p_i^2 = -k_i^2$  with  $k_i^2 > 0$ , it is then easy to show (see Appendix C) that

$$\mathbf{r}_T^2 = (1-z)k_1^2 + zk_2^2 - z(1-z)k_0^2, \quad (38)$$

where, in our notation, the fractional momentum  $z$  is simply given by  $z = z_1/(z_1 + z_2)$ . Since the two floating scales  $k_1^2$  and  $k_2^2$  can take values up to  $Q_1^2$  and  $Q_2^2$ , the maximum value of the relative transverse momentum at each branching is a function both of  $Q_1^2$  and  $Q_2^2$ . Within a leading logarithmic approximation we can set this value to be

$$Q_M^2 = \epsilon_1 Q_1^2 + \epsilon_2 Q_2^2, \quad (39)$$

with  $\epsilon_1$  and  $\epsilon_2$  being arbitrary constants of order one. This change, which is irrelevant in a leading logarithmic approximation, induces however a  $Q_2^2$  dependence in the upper integration limit of the  $\mathbf{r}_T^2$ -integral so that the inhomogeneous term in eq. (2) now reads

$$\int_{x_1}^{1-x_2} \frac{dz_1}{z_1} \int_{x_2}^{1-z_1} \frac{dz_2}{z_2} \int_{Q_0^2}^{Q_M^2} d\mathbf{r}_T^2 D_{h,corr}^{j'_1, j'_2}(z_1, z_2, \mathbf{r}_T^2) E_{j'_1}^{j_1}\left(\frac{x_1}{z_1}, \mathbf{r}_T^2, Q_1^2\right) E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, \mathbf{r}_T^2, Q_2^2\right) \Big]. \quad (40)$$

This dependence implies that the resulting DPDs evolution equations will again contain an inhomogeneous term. Since the derivation is analogous to the one presented in Ref. [3] we

just quote the result:

$$Q_2^2 \frac{\partial D_h^{j_1, j_2}(x_1, Q_1^2, x_2, Q_2^2)}{\partial Q_2^2} = \frac{\alpha_s(Q_2^2)}{2\pi} \int_{\frac{x_2}{1-x_1}}^1 \frac{du}{u} P_k^{j_2}(u) D_h^{j_1, k}(x_1, Q_1^2, x_2/u, Q_2^2) + \frac{\alpha_s(Q_2^2)}{2\pi} \frac{F_h^{j'}(x_1 + x_2, Q_2^2)}{x_1 + x_2} \widehat{P}_{j'}^{j_1, j_2}\left(\frac{x_1}{x_1 + x_2}\right), \quad (41)$$

where the initial conditions to the above evolution equations are the DPDs at  $Q_0^2$  evolved up to  $Q^2 = Q_1^2$  with the usual “equal scales” evolution equations, eq. (4). It is then easy to show that eq. (41) satisfies the momentum sum rule, eq. (6). Given the relation between relative transverse momentum and virtualities in eq. (38), the appearance of the inhomogeneous term in eq. (41) can further justified noting that it takes into account the possibility that an asymmetric configuration of virtualities could be generated in a single parton branching in the last step of the evolution. As a last remark, it is interesting to note that setting  $Q_1^2 = Q_2^2 = Q^2$  in eq. (40) it is then possible to identify with the integrand of eq. (40) the  $\mathbf{r}_T^2$ -unintegrated version of DPDs

$$\mathcal{D}_h^{j_1, j_2}(x_1, x_2, Q^2, \mathbf{r}_T^2) = \int_{x_1}^{1-x_2} \frac{dz_1}{z_1} \int_{x_2}^{1-z_1} \frac{dz_2}{z_2} D_{h, \text{corr}}^{j'_1, j'_2}(z_1, z_2, \mathbf{r}_T^2) \cdot E_{j'_1}^{j_1}\left(\frac{x_1}{z_1}, \mathbf{r}_T^2, Q^2\right) E_{j'_2}^{j_2}\left(\frac{x_2}{z_2}, \mathbf{r}_T^2, Q^2\right), \quad (42)$$

which is valid at fixed value of  $\mathbf{r}_T^2$  in the range  $Q_0^2 < \mathbf{r}_T^2 < Q^2$ . Since in eq. (42) all the  $Q^2$  dependences are contained in the  $E$ -functions, it easy to show that the corresponding evolution equations for the unintegrated  $\mathcal{D}$  are homogeneous and read

$$Q^2 \frac{\partial \mathcal{D}_h^{j_1, j_2}(x_1, x_2, Q^2, \mathbf{r}_T^2)}{\partial Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_{\frac{x_1}{1-x_2}}^1 \frac{du}{u} P_k^{j_1}(u) \mathcal{D}_h^{j_2, k}(x_1/u, x_2, Q^2, \mathbf{r}_T^2) + \frac{\alpha_s(Q^2)}{2\pi} \int_{\frac{x_2}{1-x_1}}^1 \frac{du}{u} P_k^{j_2}(u) \mathcal{D}_h^{j_1, k}(x_1, x_2/u, Q^2, \mathbf{r}_T^2). \quad (43)$$

It should be noted, however, that  $\mathbf{r}_T^2$  is not observable, at variance with the analogous case for extended dihadron [19] and fracture functions [16]. Moreover  $\mathcal{D}$  can not be readily interpreted as the distribution in relative transverse momentum of the interacting parton pair at the scale  $Q^2$  since all transverse momentum generated during the evolution up to  $Q^2$  is neglected by the  $E$ -functions, which are derived in the collinear approximation. For this interpretation to be correct one would need to replace the  $E$ -functions with appropriate Sudakov-like form factors [20, 21]. Since, in general, they tend to broaden the transverse

momentum distribution as the final scale increases [21, 22], the distribution of the relative transverse momentum at  $Q^2$  will have a broader tail with respect to the  $\mathbf{r}_T^2$  distribution. Nevertheless this result has some formal resemblance with the ones presented in Section 13 of Ref. [6] and it remains for a future task to explore whether this fact is accidental or has deeper motivations.

## V. SUMMARY

We have presented two equivalent consistency checks for the momentum sum rule for DPDs and showed the importance of the inclusion of the so called inhomogeneous term in order to obtain these results. If such term is neglected altogether, the consistency of the formalism is lost since longitudinal momentum correlations are not properly taken into account. Satisfying these consistency checks therefore impose strong constraint on the structure of DPDs evolution equations. With this respect we have revisited the result of Ref. [3] and, by a careful reexamination of the kinematics of the splitting term, we have shown the DPDs evolution equations at different virtualities does contain an inhomogeneous term.

## Appendix A

We report in this appendix the derivation of the sum rule for the second moment of the  $E$ -function introduced in eq. (8). We first take the second moment of the  $E$  evolution equations in eq. (1) which then reads

$$Q^2 \frac{\partial}{\partial Q^2} E_{i,2}^j(Q_0^2, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} A_{k,2}^j E_{i,2}^k(Q_0^2, Q^2), \quad (\text{A1})$$

and where we have introduced the Mellin transforms

$$E_{i,2}^k(Q_0^2, Q^2) = \int_0^1 dz z E_i^k(z, Q_0^2, Q^2),$$

$$A_{k,2}^j = \int_0^1 dz z P_k^j(z).$$

We may now sum eq. (A1) over the index  $j$

$$Q^2 \frac{\partial}{\partial Q^2} \sum_j E_{i,2}^j(Q_0^2, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \sum_j A_{k,2}^j E_{i,2}^k(Q_0^2, Q^2), \quad (\text{A2})$$

and exploit the following property of anomalous dimensions (the moment space analogue of eq. (29) for splitting functions)

$$\sum_j A_{k,2}^j = 0, \quad (\text{A3})$$

obtaining

$$Q^2 \frac{\partial}{\partial Q^2} \sum_j E_{i,2}^j(Q_0^2, Q^2) = 0. \quad (\text{A4})$$

This equation can be easily integrated to give

$$\sum_j E_{i,2}^j(Q_0^2, Q^2) - \sum_j E_{i,2}^j(Q_0^2, Q_0^2) = 0. \quad (\text{A5})$$

By exploiting the initial condition for the  $E$ -functions in moment space, we get the desired result

$$\sum_j E_{i,2}^j(Q_0^2, Q^2) = 1, \quad (\text{A6})$$

valid for any fixed value of the index  $i$ . It appears therefore that the moment sum rule for the  $E$ -functions is a direct consequences of eq. (A3) which, in turn, follows, for example, from conservation of overall quarks plus gluons momenta in a proton at any value of  $Q^2$ .

## Appendix B

In this appendix we explicitly check the identity quoted in the text for the particular case  $j' = q$  and  $j_2 = q$ . With this settings we have

$$\int_x^1 du (1-u) g(u) \hat{P}_q^{g,q}(1-u) = \int_x^1 du (1-u) g(u) P_q^q(u), \quad (\text{B1})$$

where the sum over  $j_1$  collapsed to just one term ( $j_1 = g$ ) due to the nature of vertex of leading order splitting functions. Substituting the relevant splitting functions

$$\hat{P}_q^{g,q}(u) = C_F \frac{1 + (1-u)^2}{u}, \quad P_q^q(u) = C_F \left( \frac{1+u^2}{1-u} \right)_+. \quad (\text{B2})$$

in eq. (B1), the latter reduces to

$$\int_x^1 du g(u) (1+u^2) = \int_x^1 du \frac{1+u^2}{1-u} [f(u) - f(1)] - f(1) \int_0^x du \frac{1+u^2}{1-u}, \quad (\text{B3})$$

where we have introduced an auxiliary function  $f(u) = (1-u)g(u)$  and exploited the standard definition of plus distribution. Since  $g(u)$  is a regular function of  $u$ , it follows

that  $f(1) = 0$  and the identity is easily proved. For the other splitting function which involves virtual contributions, namely  $P_g^g(u)$ , the calculation is analogous. Therefore, for the purpose of restoring symmetry of splitting functions upon the exchange of the daughter partons  $j_1$  and  $j_2$ , the weighting function  $(1 - u)$  is instrumental to let the virtual terms, proportional to  $\delta(1 - u)$  and at the origin of symmetry breaking, vanish.

### Appendix C

In this appendix we report the calculation used to arrive at eq. (38). See also Section 2.3 of Ref. [23]. We consider the parton branching  $0(p_0) \rightarrow 1(p_1) + 2(p_2)$  with four-momenta in parenthesis decomposed as follows

$$\begin{aligned} p_0^2 &= -k_0^2, \quad k_0^2 > 0 \\ p_1 &= zp_0 + r_T + \xi_1\eta, \\ p_2 &= (1 - z)p_0 - r_T + \xi_2\eta, \end{aligned} \tag{C1}$$

where  $\eta$  is a lightlike vector ( $\eta^2 = 0$ ) and  $r_T = (0, \mathbf{r}_T, 0)$  is the relative transverse momentum such that  $r_T^2 = -\mathbf{r}_T^2$ ,  $p_0 \cdot r_T = \eta \cdot r_T = 0$  and  $p_0 \cdot \eta \neq 0$ . The two parameters  $\xi_1$  and  $\xi_2$  can be obtained by imposing on eqs. (C1) the mass-shell relations

$$\begin{aligned} p_1^2 &= -k_1^2, \quad k_1^2 > 0, \\ p_2^2 &= -k_2^2, \quad k_2^2 > 0. \end{aligned} \tag{C2}$$

We obtain

$$\xi_1 = \frac{-k_1^2 + z^2 k_0^2 - r_T^2}{2zp_0 \cdot \eta}, \quad \xi_2 = \frac{-k_2^2 + (1 - z)^2 k_0^2 - r_T^2}{2(1 - z)p_0 \cdot \eta}. \tag{C3}$$

Squaring the momentum conservation equation,  $p_0 = p_1 + p_2$ , we get

$$2p_0 \cdot \eta(\xi_1 + \xi_2) = 0. \tag{C4}$$

Substituting the values for  $\xi_i$  and after some algebra we arrive at the desired result

$$\mathbf{r}_T^2 = (1 - z)k_1^2 + zk_2^2 - z(1 - z)k_0^2. \tag{C5}$$

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